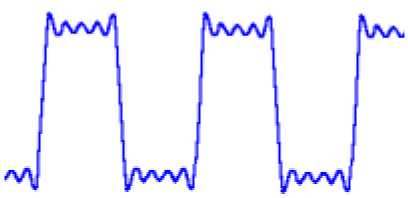
Fourier Series - Introduction

Chapter Contents

Fourier Series - Introduction

Fourier series are used in the analysis of **periodic** functions.



A periodic function

Many of the phenomena studied in engineering and science are periodic in nature eg. the current and voltage in an alternating current circuit. These periodic functions can be analysed into their constituent components (fundamentals and harmonics) by a process called **Fourier analysis**.

We are aiming to find an **approximation** using trigonometric functions for various square, saw tooth, etc waveforms that occur in electronics. We do this by adding more and more trigonometric functions together. The sum of these special trigonometric functions is called the **Fourier Series**.

Jean Fourier

Jean Baptiste Joseph **Fourier** (1768 - 1830).

Fourier was a French mathematician, who was taught by Lagrange and Laplace.

He almost died on the guillotine in the French Revolution. Fourier was a buddy of Napoleon and worked as scientific adviser for Napoleon's army.

He worked on theories of heat and expansions of functions as trigonometric series... but these were controversial at the time. Like many scientists, he had to battle to get his ideas accepted.

In this Chapter

Helpful Revision - all the trigonometry, functions, summation notation and integrals that you will need for this Fourier Series chapter.

1. Overview of Fourier Series - the definition of Fourier Series and how it is an example of a trigonometric infinite series

2. Full Range Fourier Series - various forms of the Fourier Series

3. Fourier Series of Even and Odd Functions - this section makes your life easier, because it significantly cuts down the work

4. Fourier Series of Half Range Functions - this section also makes life easier

5. Harmonic Analysis - this is an interesting application of Fourier Series

6. Line Spectrum - important in the analysis of any waveforms. Also has implications in music

7. Fast Fourier Transform - how to create CDs and how the human ear works, all with Fourier Series

We begin the chapter with Helpful Revision »

Helpful Revision for Fourier Series

This page contains some background information that will help you to better understand this chapter on Fourier Series.

You have seen most of this before, but I have included it here to give you some help before getting into the heavy stuff.

On this page :

Properties of Sine and Cosine Graphs
Periodic Functions
Continuity
Split Functions
Summation Notation
Useful Integrals

Properties of Sine and Cosine Functions

These properties can simplify the integrations that we will perform later in this chapter.

The Cosine Function

Background

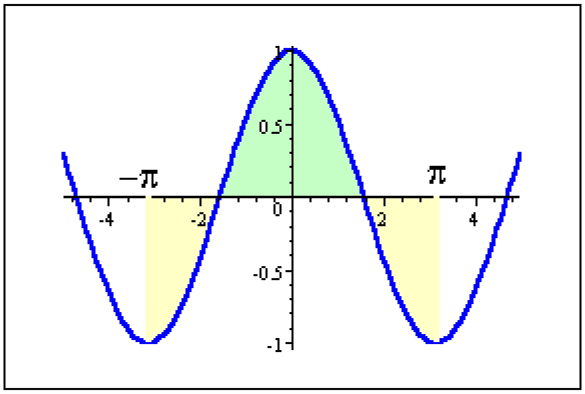
From previous chapters:

Sine and cosine curves
Even and odd functions
Integral of Sine and Cosine

The function $f(x) = \cos x$ is an **even** function. That is, it is symmetrical about the **vertical axis**.

We have: $\cos(-x) = \cos(x)$

$$\int_{-\pi}^{\pi} \cos\theta \, d\theta = 0$$

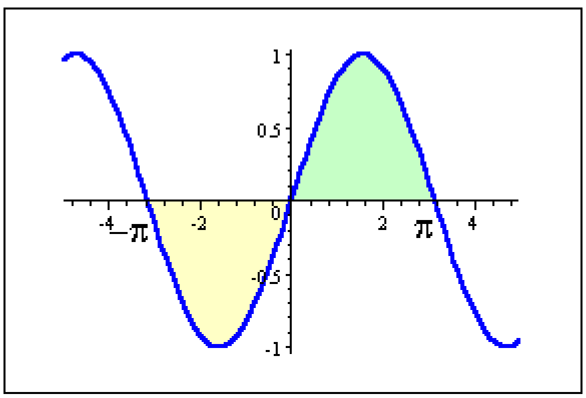


The Sine Function

The function $f(x) = \sin x$ is an **odd** function. That is, it is symmetrical about the **origin**.

We have: $\sin(-x) = -\sin(x)$

$$\int_{-\pi}^{\pi} \sin\theta \, d\theta = 0$$



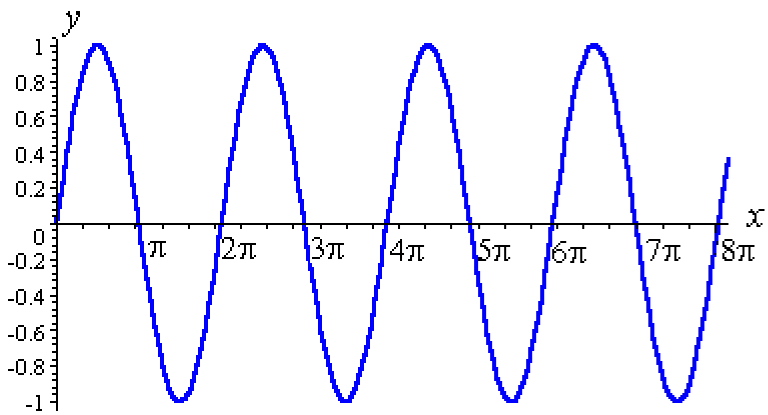
Multiples of π for Sine and Cosine Curves

Consider the function $\boldsymbol{y = \sin x}$

Revision

For some background:

[Sine and cosine curves]



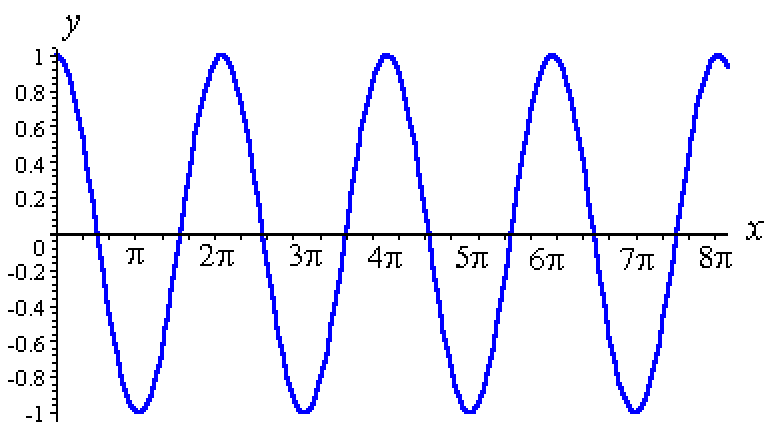
From the graph (or using our calculator), we can observe that:

$\sin(n\pi) = 0$ for $n = 0, 1, 2, 3, \ldots$ (in fact, all integers)

$\sin\frac{(2n-1)\pi}{2} = (-1)^{n+1}$ for $n = 0, 1, 2, 3, \ldots$ (in fact, all integers)

---

y* = cos *x



$\cos(2n\pi) = 1$ for $n = 0, 1, 2, 3, \ldots$ (in fact, all integers)
$\cos(2n - 1)\pi = -1$ for $n = 0, 1, 2, 3, \ldots$ (in fact, all integers)
$\cos(n\pi) = (-1)^n$ for $n = 0, 1, 2, 3, \ldots$ (in fact, all integers)

Periodic Functions

A function $f(t)$ is said to be **periodic** with **period** p if

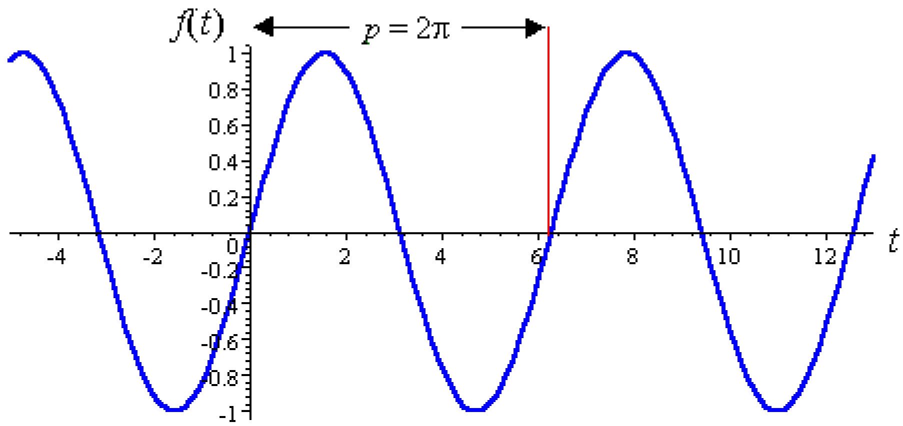
$f(t + p) = f(t)$

for all values of t and if $p > 0$.

The **period** of the function $f(t)$ is the interval between two successive repetitions.

Examples of Periodic Functions:

(a) $f(t) = \sin t$.

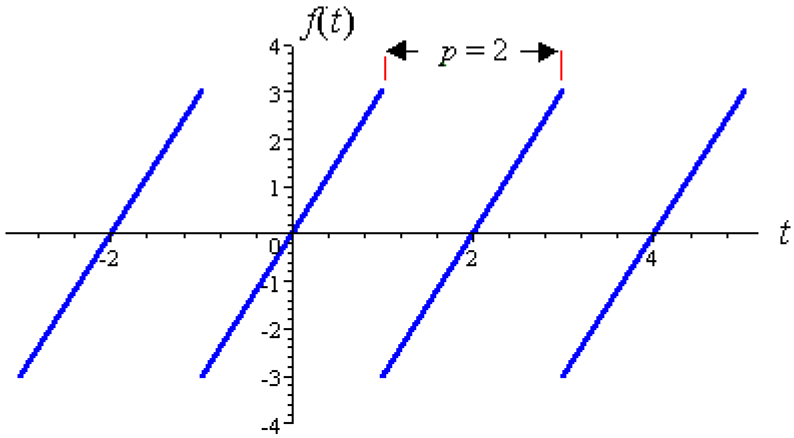


Useful Background

[Sine and cosine curves]

For $f(t) = \sin t$, we have: $f(t) = f(t + 2\pi)$. The period is 2π.

(b) Saw tooth waveform, period = 2:



Useful background

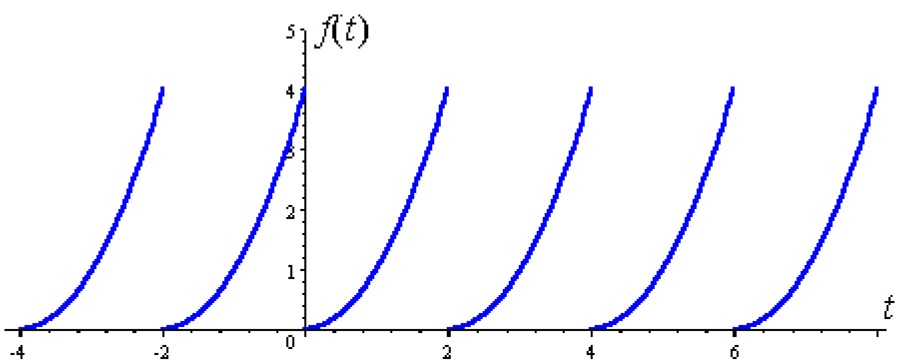
Straight lines

For this function, we have:

$f(t) = 3t$ (for $-1 \leq t < 1$)

$f(t) = f(t + 2)$ [This indicates it is periodic with period 2.]

(c) Parabolic, period = 2.



Useful background

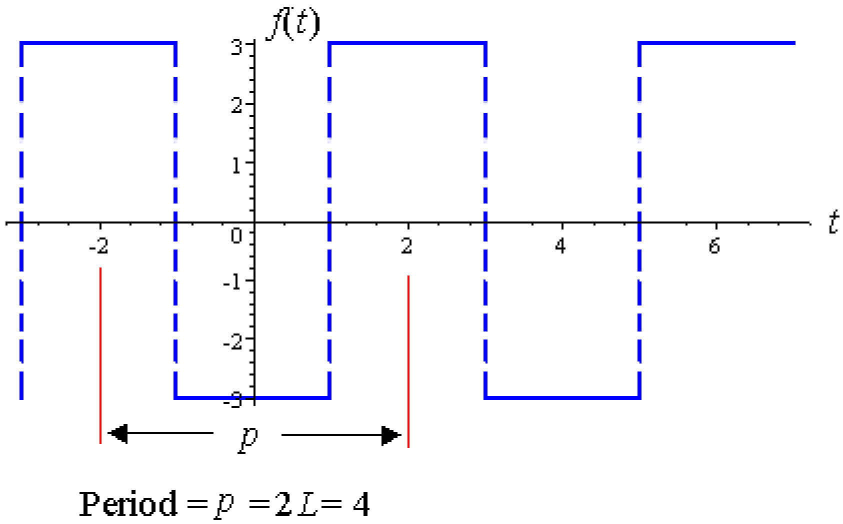
Parabolas

For this function, we have:

$f(t) = t^2$ (for $0 \leq t < 2$)

$f(t) = f(t + 2)$ [Indicating it is periodic with period 2.]

(d) Square wave, period = 4.



For this function, we have:

$f(t) = -3$ for $-1 \le t < 1$ and 3 for $1 \le t < 3$

$f(t) = f(t + 4)$ [The period is 4.]

NOTE: In this example, the period $p = 4$. We can write this as $2L = 4$.

In the diagram we are thinking of one cycle starting at −2 and finishing at 2. For convenience when integrating later, we let $L = 2$ and the cycle goes from $-L$ to L.

Continuity

If a graph of a function has no sudden jumps or breaks, it is called a **continuous** function.

Examples:

Useful Background

Continuous Functions
Exponential graphs
Parabolas

- sine functions
- cosine functions
- exponential functions
- parabolic functions

Finite discontinuity - a function makes a finite jump at some point or points in the interval.

Examples:

- Square wave function
- Saw tooth functions

Split Functions

Much of the behaviour of current, charge and voltage in an AC circuit can be described using split functions.

Examples of Split Functions

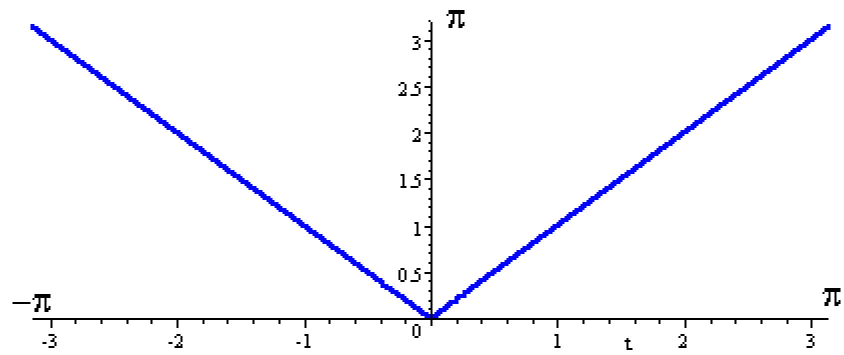
Sketch the following functions:

Useful Background

Split Functions
Straight lines

(a) $$f(t) = \begin{cases} -t & \text{if } -\pi \le t < 0 \\ t & \text{if } 0 \le t < \pi \end{cases}$$

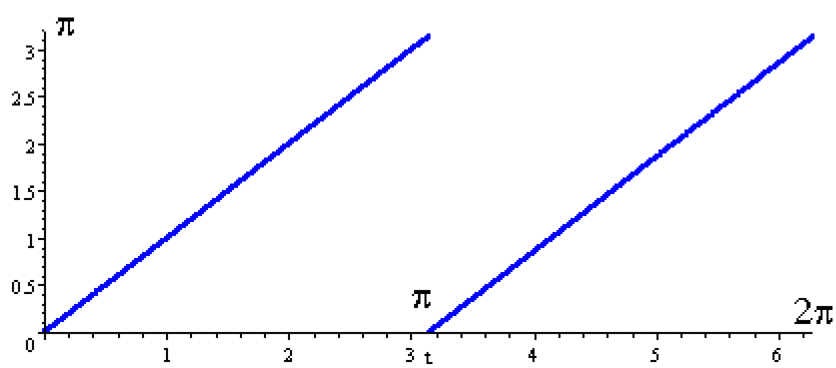
Answer



(b)

$$f(t) = \begin{cases} t & \text{if} \quad 0 \le t < \pi \\ t - \pi & \text{if} \quad \pi \le t < 2\pi \end{cases}$$

Answer



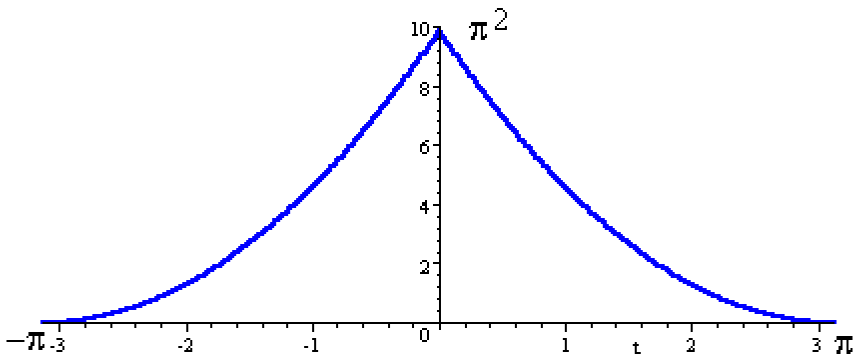
(c)

$$f(t) = \begin{cases} (t+\pi)^2 & \text{if} \quad -\pi \le t < 0 \\ (t-\pi)^2 & \text{if} \quad 0 \le t < \pi \end{cases}$$

Useful Background

Parabolas

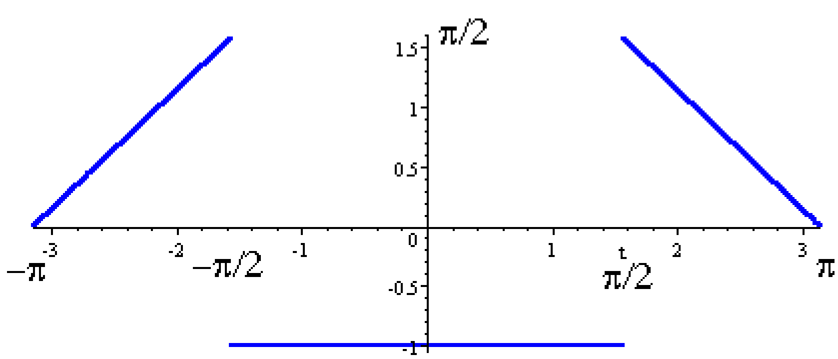
Answer



(d)

$$f(t) = \begin{cases} t + \pi & \text{if } -\pi \le t < -\frac{\pi}{2} \\ -1 & \text{if } -\frac{\pi}{2} \le t < \frac{\pi}{2} \\ -t + \pi & \frac{\pi}{2} \le t < \pi \end{cases}$$

Answer



Summation Notation

It is important to understand **summation notation** when dealing with Fourier series.

Examples

Expand the following and simplify where possible:

1. $\sum_{n=1}^{3} \frac{n}{n+1}$

Answer

$$\sum_{n=1}^{3} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$$

2. $\sum_{n=1}^{5} (2n-1)$

Answer

$$\sum_{n=1}^{5} (2n-1) = 1 + 3 + 5 + 7 + 9 = 25$$

Notice that ($2n$ - 1) generates **odd numbers.**

If we want to generate **even numbers**, we would use $\boldsymbol{2n}$.

To generate **alternate positive and negative numbers**, use $(-1)^{n+1}$.

So $\frac{(-1)^n}{2n}$ generates 1/2, -1/4, 1/6, -1/8, ...

3. $\sum_{n=1}^{5} n^2 a_n$

Answer

$$\sum_{n=1}^{5} n^2 a_n = a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5$$

4. $\displaystyle\sum_{n=1}^{4} \frac{n\pi t}{L}$

Answer

$$\begin{aligned}\sum_{n=1}^{4} \frac{n\pi t}{L} &= \frac{\pi t}{L} + \frac{2\pi t}{L} + \frac{3\pi t}{L} + \frac{4\pi t}{L} \\ &= \frac{\pi t}{L}(1+2+3+4) \\ &= 10\frac{\pi t}{L}\end{aligned}$$

Some Useful Integrals

These are obtained from **integration by parts:**

$$\int t \sin nt\,dt = \frac{1}{n^2}(\sin nt - nt\cos nt)$$

$$\int t \cos nt\,dt = \frac{1}{n^2}(\cos nt + nt\sin nt)$$

1. Overview of Fourier Series

In mathematics, infinite series are very important. They are used extensively in calculators and computers for evaluating values of many functions.

The Fourier Series is really interesting, as it uses many of the mathematical techniques that you have learned before, like graphs, integration, differentiation, summation notation, trigonometry, etc. If you get stuck, see if the Helpful Revision page gives you inspiration.

Infinite Series - Numbers

Useful Background

Check out the Series chapter, especially Infinite series. (In particular, note what it says about **convergence** of an infinite series.)

A **geometric progression** is a set of numbers with a common ratio.

Example: 1, 2, 4, 8, 16

A **series** is the **sum** of a sequence of numbers.

Example: 1 + 2 + 4 + 8 + 16

An **infinite series** that converges to a particular value has a common ratio less than 1.

Example: 1 + 1/3 + 1/9 + 1/27 + ... = 3/2

When we expand functions in terms of some infinite series, the series will **converge** to the function as we take more and more terms.

Infinite Series Expansions of Functions

We learned before in the Infinite Series Expansions chapter how to re-express many functions (like $\sin x$, $\log x$, e^x, etc) as a polynomial with an infinite number of terms.

We saw how our polynomial was a good approximation near some value $x = a$ (in the case of Taylor Series) or $x = 0$ (in the case of Maclaurin Series). To get a better approximation, we needed to add more terms of the polynomial.

Fourier Series - A Trigonometric Infinite Series

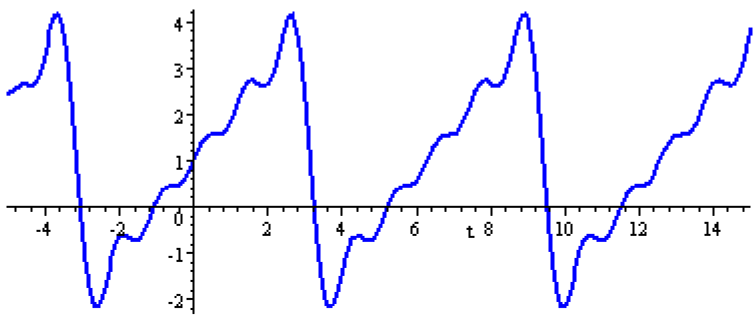
In this chapter we are also going to re-express functions in terms of an infinite series. However, instead of using a polynomial for our infinite series, we are going to use the sum of **sine** and **cosine** functions.

Fourier Series is used in the analysis of signals in electronics. For example, later we will see the Fast Fourier Transform, which talks about pulse code modulation which is used when recording digital music.

Example

We will see functions like the following, which approximates a saw-tooth signal:

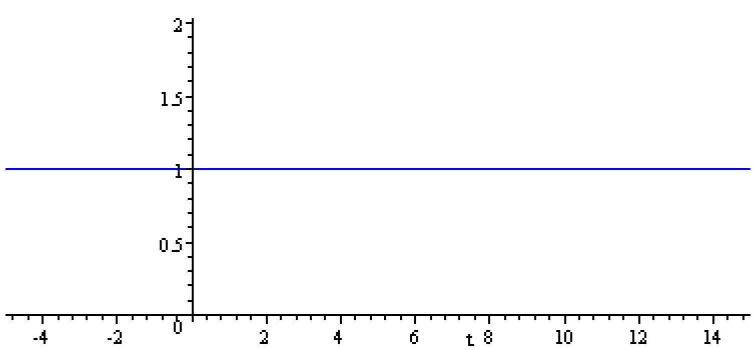
$$f(x) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t + \ldots$$



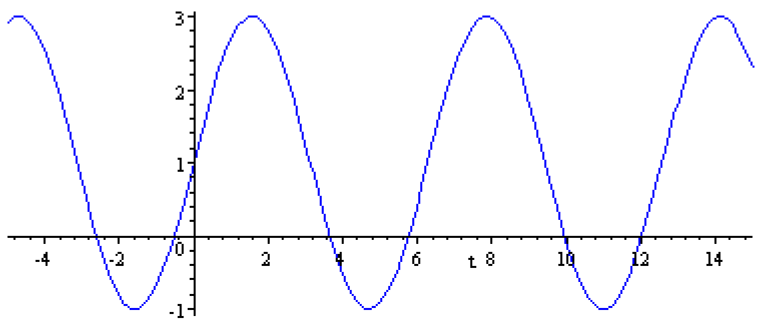
How does it work? As we add more terms to the series, we find that it converges to a particular shape.

Taking one extra term in the series each time and drawing separate graphs, we have:

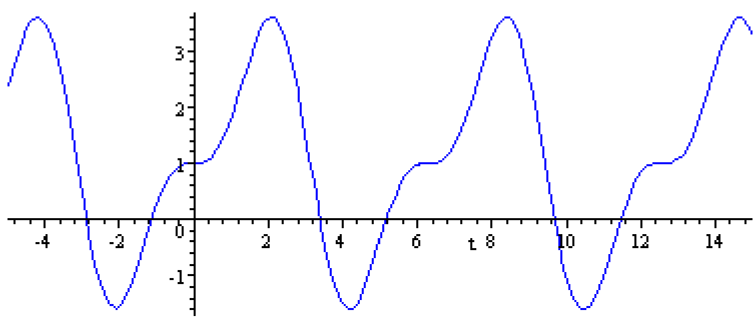
$f(t) = 1$ (first term of the series):



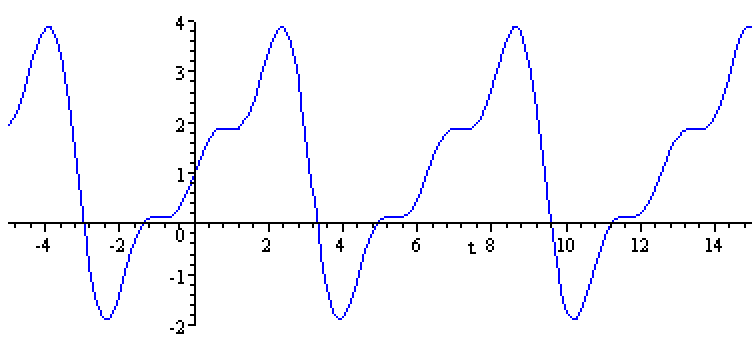
$f(t) = 1 + 2\sin t$ (first 2 terms of the series):



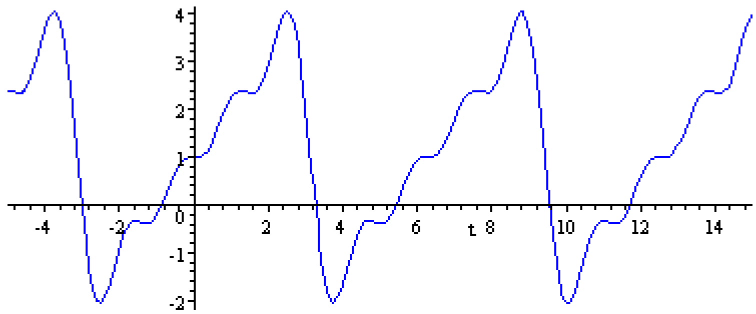
$f(t) = 1 + 2\sin t - \sin 2t$ (first 3 terms of the series):



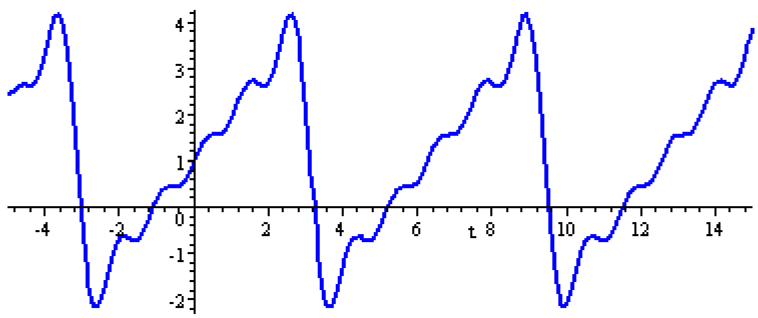
$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t$$



$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$$



$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t$$



We say that the infinite Fourier series **converges** to the saw tooth curve.

That is, if we take more and more terms, the graph will look more and more like a saw tooth. If we could take an **infinite** number of terms, the graph would look like a set of saw teeth...

We will see how this works, and where the terms in the series come from, in the next sections.

2. Full Range Fourier Series

The Fourier Series is an infinite series expansion involving trigonometric functions.

A periodic waveform $f(t)$ of period $p = 2L$ has a Fourier Series given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi t}{L} + a_2 \cos\frac{2\pi t}{L} + a_3 \cos\frac{3\pi t}{L} + \ldots$$

$$+ b_1 \sin\frac{\pi t}{L} + b_2 \sin\frac{2\pi t}{L} + b_3 \sin\frac{3\pi t}{L} + \ldots$$

Helpful Revision

Summation Notation (∑)

where

a_n and b_n are the **Fourier coefficients**,

and

$\frac{a_0}{2}$ is the **mean value**, sometimes referred to as the **dc level**.

Fourier Coefficients For Full Range Series Over Any Range -*L* TO *L*

If $f(t)$ is expanded in the range $-L$ to L (period = $2L$) so that the range of integration is $2L$, i.e. half the range of integration is L, then the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

where $n = 1, 2, 3 \ldots$

NOTE: Some textbooks use

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$

and then modify the series appropriately. It gives us the same final result.

Dirichlet Conditions

Any periodic waveform of period $p = 2L$, can be expressed in a Fourier series provided that

(a) it has a finite number of discontinuities within the period $2L$;

(b) it has a finite average value in the period $2L$;

(c) it has a finite number of positive and negative maxima and minima.

When these conditions, called the Dirichlet conditions, are satisfied, the Fourier series for the function $f(t)$ exists.

Each of the examples in this chapter obey the Dirichlet Conditions and so the Fourier Series exists.

Example of a Fourier Series - Square Wave

Sketch the function for 3 cycles:

$$f(t) = \begin{cases} 0 & \text{if } -4 \le t < 0 \\ 5 & \text{if } 0 \le t < 4 \end{cases}$$

$f(t) = f(t + 8)$

Find the Fourier series for the function.

Solution:

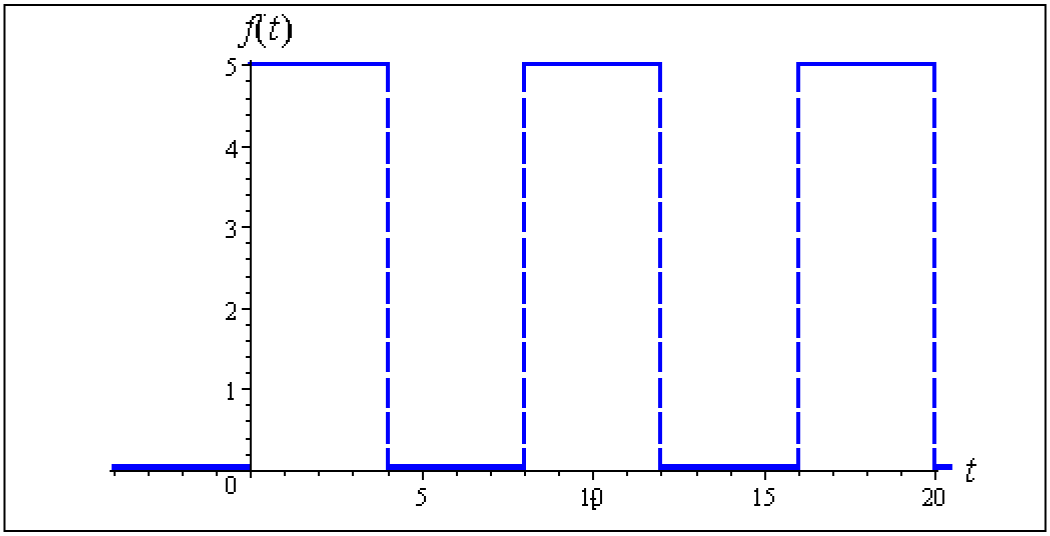
First, let's see what we are trying to do by seeing the final answer using a LiveMath animation.

LIVEMath

Now for one possible way to solve it:

Answer

The sketch of the function:



We need to find the **Fourier coefficients** a_0, a_n and b_n before we can determine the series.

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)dt \\ &= \frac{1}{4}\int_{-4}^{4} f(t)dt \\ &= \frac{1}{4}\left(\int_{-4}^{0}(0)dt + \int_{0}^{4}(5)dt\right) \\ &= \frac{1}{4}\left(0 + [5t]_0^4\right) \\ &= \frac{1}{4}(20) \\ &= 5 \end{aligned}$$

Note 1: We could have found this value easily by observing that the graph is totally above the t-axis and finding the area under the curve from $t = 4$ to $t = 4$. It is just 2 rectangles, one with height 0 so the area is 0, and the other rectangle has dimensions 4 by 5, so the area is 20. So the integral part has value 20; and 1/4 of 20 = 5.

Note 2: The mean value of our function is given by $a_0/2$. Our function has value 5 for half of the time and value 0 for the other half, so the value of $a_0/2$ must be 2.5. So a_0 will have value 5.

These points can help us check our work and help us understand what is going on. However, it is good to see how the integration works for a split function like this.

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\
&= \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}\,dt \\
&= \frac{1}{4}\left(\int_{-4}^{0}(0)\cos\frac{n\pi t}{4}\,dt + \int_{0}^{4}(5)\cos\frac{n\pi t}{4}\,dt\right) \\
&= \frac{1}{4}\left(0 + \left[5\frac{4}{n\pi}\sin\frac{n\pi t}{4}\right]_0^4\right) \\
&= \frac{1}{4}\times 5\times\frac{4}{n\pi}\left(\left[\sin\frac{n\pi t}{4}\right]_0^4\right) \\
&= \frac{5}{n\pi}\left(\sin\frac{n\pi(4)}{4} - \sin\frac{n\pi(0)}{4}\right) \\
&= \frac{5}{n\pi}(\sin n\pi - 0) \\
&= 0
\end{aligned}$$

Note: In the next section, Even and Odd Functions, we'll see that we don't even need to calculate a_n in this example. We can tell it will have value 0 before we start.

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \\
&= \frac{1}{4}\int_{-4}^{4} f(t)\sin\frac{n\pi t}{4}\,dt \\
&= \frac{1}{4}\left(\int_{-4}^{0}(0)\sin\frac{n\pi t}{4}\,dt + \int_{0}^{4}(5)\sin\frac{n\pi t}{4}\,dt\right) \\
&= \frac{1}{4}\left(0 + \left[-5\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right]_0^4\right) \\
&= -\frac{1}{4}\times 5\times\frac{4}{n\pi}\left[\cos\frac{n\pi t}{4}\right]_0^4 \\
&= -\frac{5}{n\pi}\left(\cos\frac{n\pi(4)}{4} - 1\right) \\
&= -\frac{5}{n\pi}(\cos n\pi - 1)
\end{aligned}
$$

At this point, we can substitute this into our Fourier Series formula:

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi t}{L} \\
&= \frac{5}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi t}{4} + \sum_{n=1}^{\infty} -\frac{5}{n\pi}(\cos n\pi - 1)\sin\frac{n\pi t}{4} \\
&= 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}
\end{aligned}
$$

Now, we substitute $n = 1, 2, 3,...$ into the expression inside the series:

n	$\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$
1	$\frac{1}{1}(\cos\pi - 1)\sin\frac{\pi t}{4} = -2\sin\frac{\pi t}{4}$
2	$\frac{1}{2}(\cos 2\pi - 1)\sin\frac{2\pi t}{4} = 0$
3	$\frac{1}{3}(\cos 3\pi - 1)\sin\frac{3\pi t}{4} = -\frac{2}{3}\sin\frac{3\pi t}{4}$
4	$\frac{1}{4}(\cos 4\pi - 1)\sin\frac{4\pi t}{4} = 0$
5	$\frac{1}{5}(\cos 5\pi - 1)\sin\frac{5\pi t}{4} = -\frac{2}{5}\sin\frac{5\pi t}{4}$
6	$\frac{1}{6}(\cos 6\pi - 1)\sin\frac{6\pi t}{4} = 0$
7	$\frac{1}{7}(\cos 7\pi - 1)\sin\frac{7\pi t}{4} = -\frac{2}{7}\sin\frac{7\pi t}{4}$

Now we can write out the first few terms of the required Fourier Series:

$$
\begin{aligned}
f(t) &= 2.5 - \frac{5}{\pi}\left(-2\sin\frac{\pi t}{4} - \frac{2}{3}\sin\frac{3\pi t}{4} - \frac{2}{5}\sin\frac{5\pi t}{4} - \dots\right) \\
&= 2.5 + \frac{10}{\pi}\left(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4} + \frac{1}{5}\sin\frac{5\pi t}{4} + \dots\right)
\end{aligned}
$$

Alternative approach:

Answer

Alternatively, we could observe that every even term is 0, so we only need to generate odd terms. We could have expressed the b_n term as:

$$b_n = -\frac{5}{n\pi}(\cos n\pi - 1)$$

$$= -\frac{5}{n\pi}((-1)^n - 1)$$

$$= \frac{10}{n\pi} \qquad n \text{ odd, (0 if } n \text{ is even)}$$

To generate odd numbers for our series, we need to use:

$$b_n = \frac{10}{(2n-1)\pi} \qquad n = 1, 2, 3\ldots$$

We also need to generate only odd numbers for the sine terms in the series, since the even ones will be 0.

So the required series this time is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{5}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi t}{4} + \sum_{n=1}^{\infty}\frac{10}{(2n-1)\pi}\sin\frac{(2n-1)\pi t}{4}$$

$$= 2.5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi t}{4}$$

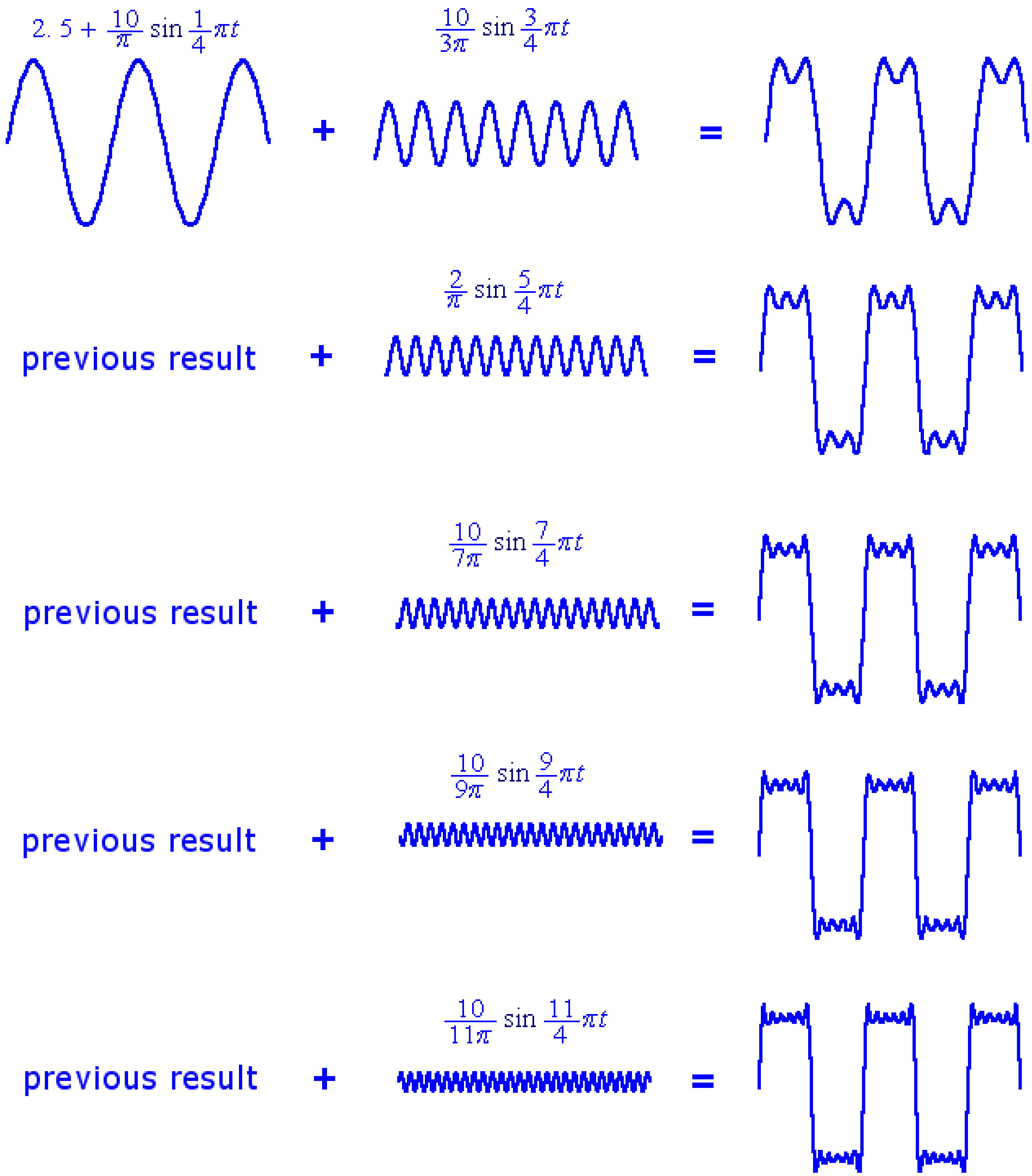
The first four terms series are once again:

$$f(t) = 2.5 + \frac{10}{\pi}\left(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4} + \frac{1}{5}\sin\frac{5\pi t}{4} + \frac{1}{7}\sin\frac{7\pi t}{4} + \ldots\right)$$

[**NOTE:** Whichever method we choose, *n* must take values 1, 2, 3, ... when we are writing out the series using sigma notation.]

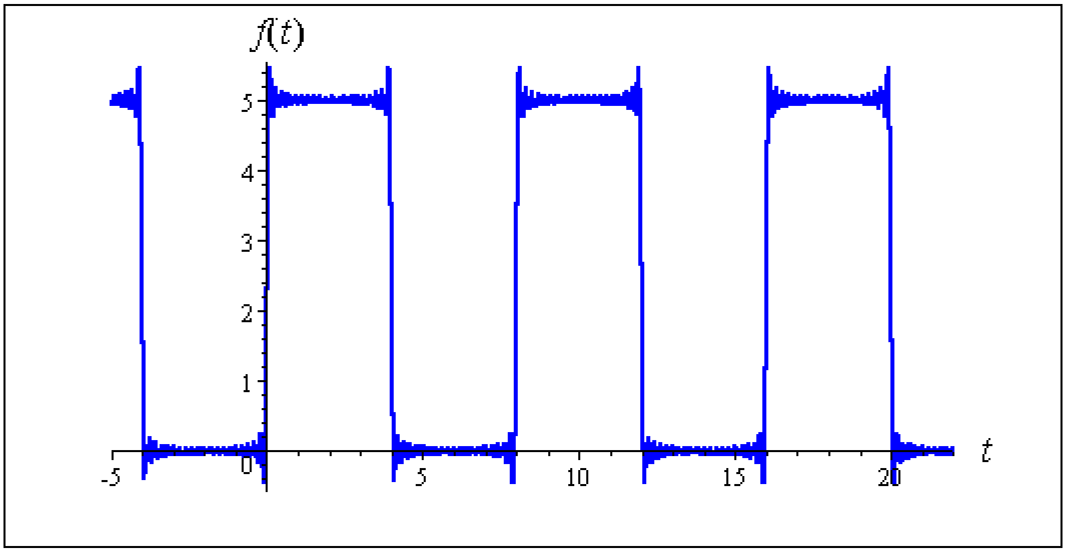
What have we done?

We are adding a series of sine terms (with decreasing amplitudes and decreasing periods) together. The combined signal, as we take more and more terms, starts to look like our original square wave:



If we graph many terms, we see that our series is producing the required function. We graph the first 20 terms:

$$2.5+\frac{10}{\pi}\sum_{n=1}^{20}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi t}{4}$$



Apart from helping us understand what we are doing, a graph can help us check our calculations...

Common Case: Period = 2*L*= 2π

If a function is defined in the range -π to π (i.e. period $2L = 2\pi$ radians), the range of integration is 2π and half the range is $L = \pi$.

The Fourier coefficients of the Fourier series $f(t)$ in this case become:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

and the formula for the Fourier Series becomes:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where $n = 1, 2, 3, \ldots$

Example

a) Sketch the waveform of the periodic function defined as:

$f(t) = t$ for $-\pi < t < \pi$

$f(t) = f(t + 2\pi)$ for all t.

b) Obtain the Fourier series of $f(t)$ and write the first 4 terms of the series.

Answer

Loading...

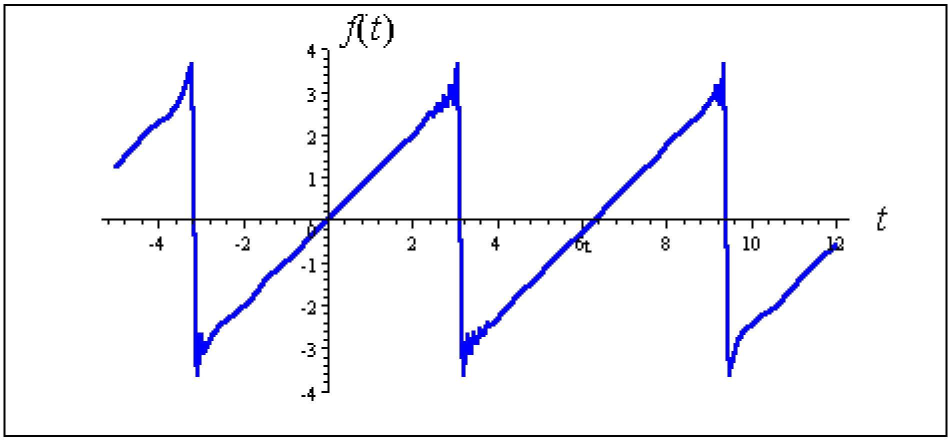
What have we found?

Let's see an animation of this example using LiveMath.

LIVEMath

The graph of the first 40 terms is:

$$\sum_{n=1}^{40} \left(\frac{2}{n} (-1)^{n+1} \sin nt \right)$$



We can express the Fourier Series in different ways for convenience, depending on the situation.

Fourier Series Expanded In Time *t* with period *T*

Let the function $f(t)$ be periodic with period $T = 2L$ where

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}.$$

In this case, our lower limit of integration is 0.

Hence the Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)dt$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,dt \quad b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt$$

(Note: half the range of integration = π/ω)

Fourier Series Expanded in Angular Displacement ω

(**Note:** ω is measured in radians here)

Let the function $f(\omega)$ be periodic with period $2L$.

We let $\theta = \omega t$. This function can be represented as

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi\theta}{L} + b_n \sin\frac{n\pi\theta}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^{2L} f(\theta)d\theta$$

$$a_n = \frac{1}{L}\int_0^{2L} f(\theta)\cos\frac{n\pi\theta}{L}d\theta \quad b_n = \frac{1}{L}\int_0^{2L} f(\theta)\sin\frac{n\pi\theta}{L}d\theta$$

3. Fourier Series of Even and Odd Functions

This section can make our lives a lot easier because it reduces the work required.

In some of the problems that we encounter, the Fourier coefficients a_o, a_n or b_n become **zero** after integration.

Revision

Go back to Even and Odd Functions for more information.

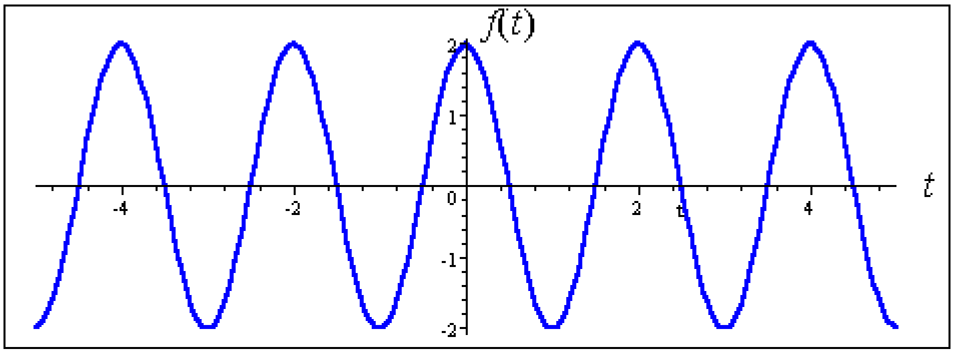
Finding zero coefficients in such problems is time consuming and can be avoided. With knowledge of **even and odd functions**, a zero coefficient may be predicted without performing the integration.

Even Functions

Recall: A function $y = f(t)$ is said to be **even** if $f(-t) = f(t)$ for all values of t. The graph of an **even** function is always symmetrical about the ***y*-axis** (i.e. it is a mirror image).

Example of an Even Function

$f(t) = 2 \cos \pi t$



Fourier Series for Even Functions

For an **even** function $f(t)$, defined over the range $-L$ to L (i.e. period = $2L$), we have the following handy short cut.

Since

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt$$

and

$f(t)$ is even,

it means the integral will have value 0. (See Properties of Sine and Cosine Graphs.)

So for the Fourier Series for an even function, the coefficient $\boldsymbol{b_n}$ has zero value:

$\boldsymbol{b_n = 0}$

So we only need to calculate a_0 and a_n when finding the Fourier Series expansion for an even function $f(t)$:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t) dt \quad a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt$$

An **even** function has only **cosine** terms in its Fourier expansion:

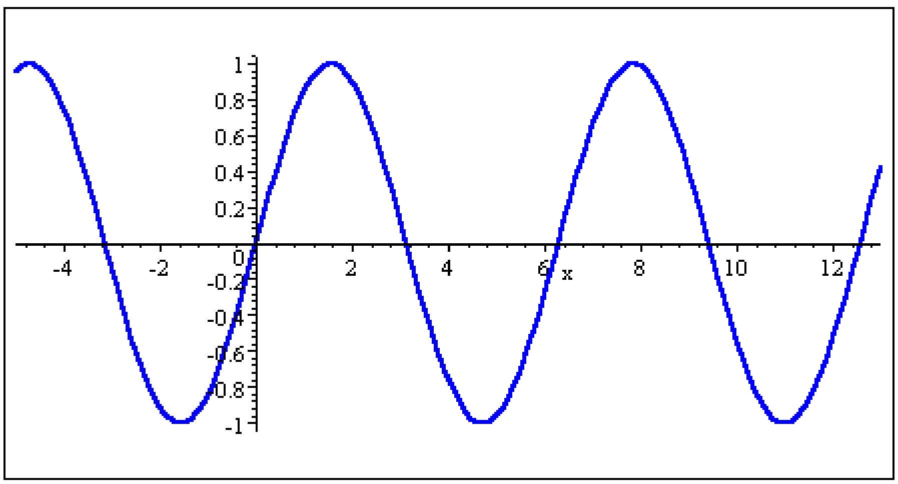
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

Fourier Series for Odd Functions

Recall: A function $y = f(t)$ is said to be **odd** if $f(-t) = -f(t)$ for all values of t. The graph of an **odd** function is always symmetrical about the **origin**.

Example of an Odd Function

$f(t) = \sin t$



Fourier Series for Odd Functions

For an **odd** function $f(t)$ defined over the range $-L$ to L (i.e. period = $2L$), we find that $a_n = 0$ for all n.

Since

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

The **zero coefficients** in this case are: $\boldsymbol{a_0 = 0}$ and $\boldsymbol{a_n = 0}$.

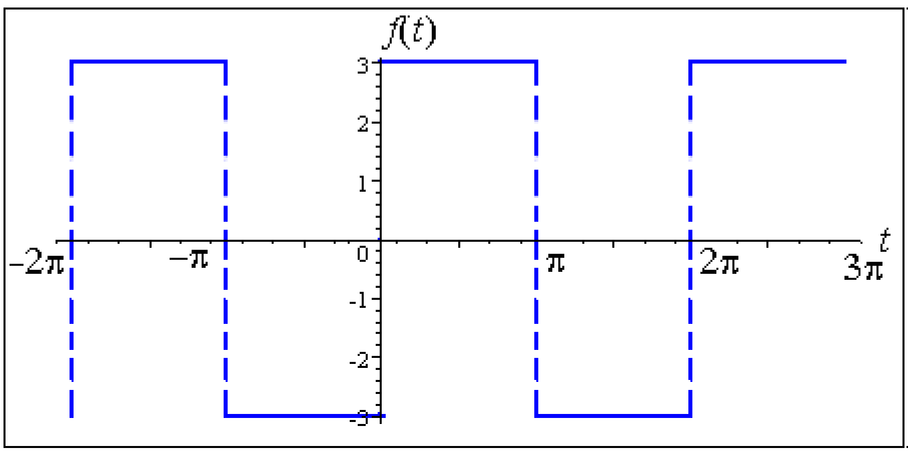
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

An odd function has only **sine** terms in its Fourier expansion.

Exercise 1

Find the Fourier Series for the function for which the graph is given by:



Answer

First, we need to define the function:

$$f(t) = \begin{cases} -3 & \text{if } -\pi \le t < 0 \\ 3 & \text{if } 0 \le t < \pi \end{cases}$$

We can see from the graph that it is periodic, with period 2π.
So $f(t) = f(t + 2\pi)$.

Also, $L = \pi$.

We can also see that it is an **odd** function, so we know $a_0 = 0$ and $a_n = 0$. So we will only need to find b_n.

Since $L = \pi$, the necessary formulae become:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

Now

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -3\sin nt\,dt + \int_{0}^{\pi} 3\sin nt\,dt\right) \\ &= \frac{3}{\pi}\left(\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nt}{n}\right]_{0}^{\pi}\right) \\ &= \frac{3}{\pi n}(\cos 0 - \cos(-\pi n) - \cos n\pi + \cos 0) \\ &= \frac{3}{\pi n}(2 - \cos \pi n - \cos \pi n) \\ &= \frac{6}{\pi n}(1 - \cos \pi n) \\ &= \frac{12}{\pi n} \qquad n \text{ odd} \\ &= 0 \qquad n \text{ even} \end{aligned}$$

We could write this as: $$b_n = \frac{12}{\pi(2n-1)} \qquad n = 1, 2, 3\ldots$$

So the Fourier series for our odd function is given by:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

$$= \sum_{n=1}^{\infty} \frac{12}{\pi(2n-1)} \sin(2n-1)t$$

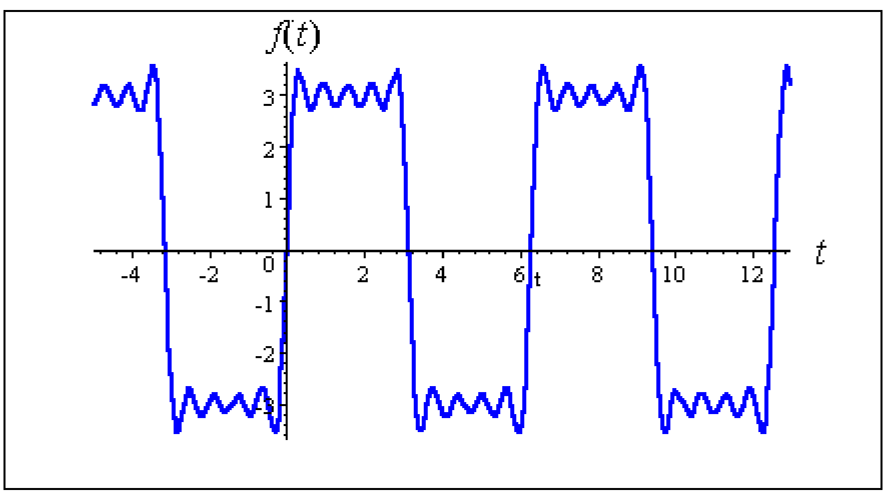
$$= \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)}$$

NOTE: Since b_n is non-zero for n odd, we must also have odd multiples of t within the sine expression (the even ones are multiplied by 0, so will be 0).

Checking, we take the first 5 terms:

$$\frac{12}{\pi} \sum_{n=1}^{5} \frac{\sin(2n-1)t}{(2n-1)}$$

$$= \frac{12}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \frac{1}{9} \sin 9t \right)$$



We see that the graph of the first 5 terms is certainly approaching the shape of the graph that was in the question. We can be confident we have the correct answer.

Exercise 2

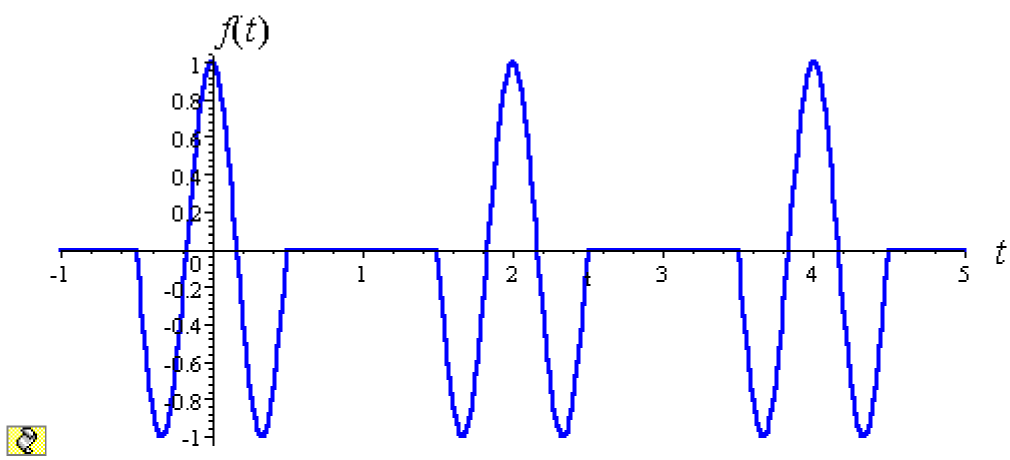
Sketch 3 cycles of the function represented by

$$f(t) = \begin{cases} 0 & \text{if } -1 \le t < -\frac{1}{2} \\ \cos 3\pi t & \text{if } -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le t < 1 \end{cases}$$

and $f(t) = f(t + 2)$.

Find the Fourier Series.

Answer



This function is an **even function**, so $b_n = 0$. We only need to find $\boldsymbol{a_0}$ and $\boldsymbol{a_n}$ **.**

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)dt \\
&= \int_{-1}^{1} f(t)dt \\
&= \int_{-1}^{-0.5} 0dt + \int_{-0.5}^{0.5} \cos 3\pi t\, dt + \int_{0.5}^{1} 0dt \\
&= \left[\frac{\sin 3\pi t}{3\pi}\right]_{-0.5}^{0.5} \\
&= -\frac{2}{3\pi}
\end{aligned}
$$

Now for a_n. We will use Scientific Notebook to perform the integration:

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt \\
&= \int_{-1}^{1} f(t)\cos n\pi t\, dt \\
&= 0 + \int_{-0.5}^{0.5} \cos 3\pi t \cos n\pi t\, dt + 0 \\
&= 6\frac{\cos\frac{1}{2}n\pi}{\pi(-9+n^2)} \quad \text{using Scientific Notebook}
\end{aligned}
$$

Recall that cos($n\pi/2$) = 0 for n odd and +1 or -1 for n even. So we expect 0 for every odd term.

However, we cannot have n = 3 in this expression, since the denominator would be 0. In this situation, we need to integrate for n = 3 to see if there is a value. In fact, we will use SNB to find the values up to n = 5, to see what is happening:

When $n = 1$, $\int_{-1/2}^{1/2} \cos 3\pi t \cos \pi t dt = 0$

When $n = 2$, $\int_{-1/2}^{1/2} \cos 3\pi t \cos 2\pi t dt = \frac{6}{5\pi}$

When $n = 3$, $\int_{-1/2}^{1/2} \cos 3\pi t \cos 3\pi t dt = \frac{1}{2}$

When $n = 4$, $\int_{-1/2}^{1/2} \cos 3\pi t \cos 4\pi t dt = \frac{6}{7\pi}$

When $n = 5$, $\int_{-1/2}^{1/2} \cos 3\pi t \cos 5\pi t dt = 0$

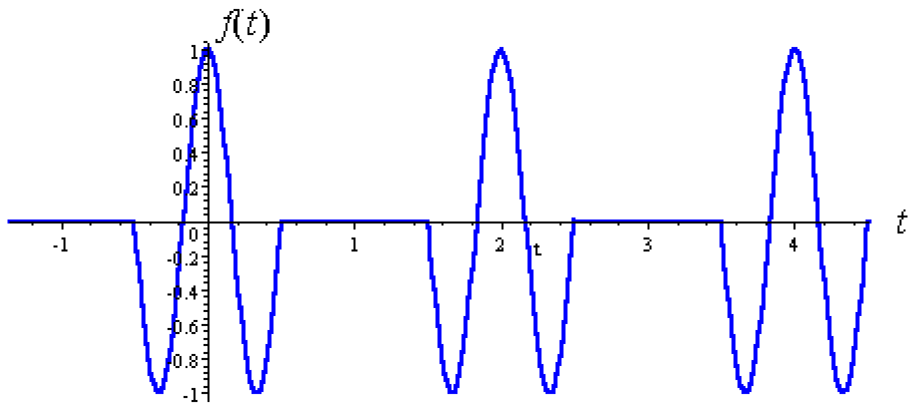
So we will start our series by writing out the terms for $n = 2$ and $n = 3$, then use summation notation from $n = 4$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

$$= \frac{a_0}{2} + \frac{6}{5\pi} \cos 2\pi t + \frac{1}{2} \cos 3\pi t + \sum_{n=4}^{\infty} a_n \cos \frac{n\pi t}{L}$$

$$= -\frac{1}{3\pi} + \frac{6 \cos 2\pi t}{5\pi} + \frac{\cos 3\pi t}{2} + \sum_{n=4}^{\infty} 6 \frac{\cos \frac{n\pi}{2}}{\pi(-9 + n^2)} \cos n\pi t$$

As usual, we graph the first few terms and see that our series is correct:



Solution without Scientific Notebook:

The integration for a_n could have been performed as follows. We re-express the function using a trick based on what we learned in Sum and Difference of Two Angles.

$$\cos 3\pi t \cos n\pi t$$

$$= \frac{1}{2}(\cos(1+n)3\pi t + \cos(1-n)3\pi t)$$

$$\int_{-1/2}^{1/2} \cos 3\pi t \cos n\pi t dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2}(\cos(1+n)3\pi t + \cos(1-n)3\pi t)dt$$

$$= \frac{1}{2}\left[\frac{\sin(1+n)3\pi t}{(1+n)3\pi} + \frac{\sin(1-n)3\pi t}{(1-n)3\pi}\right]_{-1/2}^{1/2}$$

It is then necessary to substitute $t = 1/2$ and $t = -1/2$ as usual, then simplify the expression in n.

After integrating, we could have expressed a_n as follows:

$$a_n = 6\frac{(-1)^{n/2}}{\pi(-9+n^2)} \qquad n \text{ even} \quad (0 \text{ if } n \text{ odd}, n \neq 3)$$

$$= 6\frac{(-1)^n}{\pi(-9+4n^2)} \qquad n = 1, 2, 3, \ldots$$

Then we could have substituted this expression into the series. However, we would still need to consider separately the case when $n = 3$.

4. Half Range Fourier Series

If a function is defined over half the range, say 0 to L, instead of the full range from $-L$ to L, it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a **half range Fourier series**.

Conversely, the Fourier Series of an even or odd function can be analysed using the half range definition.

Even Function and Half Range Cosine Series

An **even** function can be expanded using half its range from

- 0 to L or
- $-L$ to 0 or
- L to $2L$

That is, the range of integration = L. The Fourier series of the **half range even function** is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

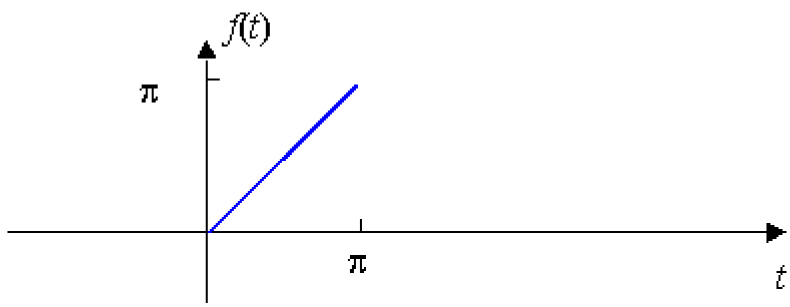
for $n = 1, 2, 3, \ldots$, where

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

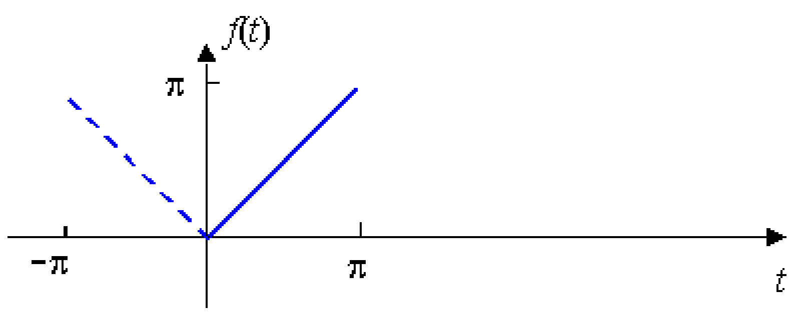
$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

$b_n = 0$

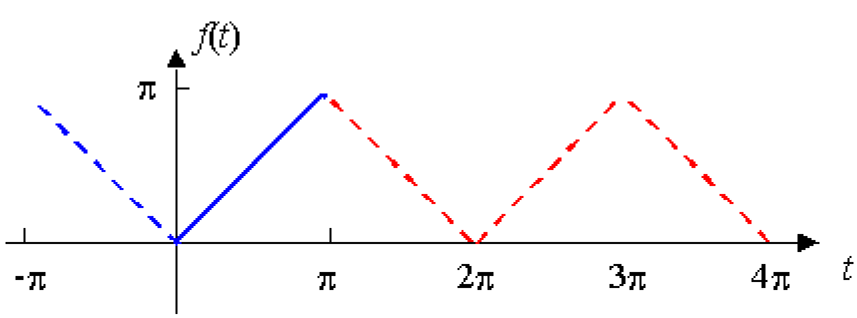
In the figure below, $f(t) = t$ is sketched from $t = 0$ to $t = \pi$.



An even function means that it must be symmetrical about the $f(t)$ axis and this is shown in the following figure by the broken line between $t = -\pi$ and $t = 0$.



It is then assumed that the "triangular wave form" produced is periodic with period 2π outside of this range as shown by the red dotted lines.



Example

We are given that

$$f(t) = \begin{cases} -t & \text{if } -\pi \le t < 0 \\ t & \text{if } 0 \le t < \pi \end{cases}$$

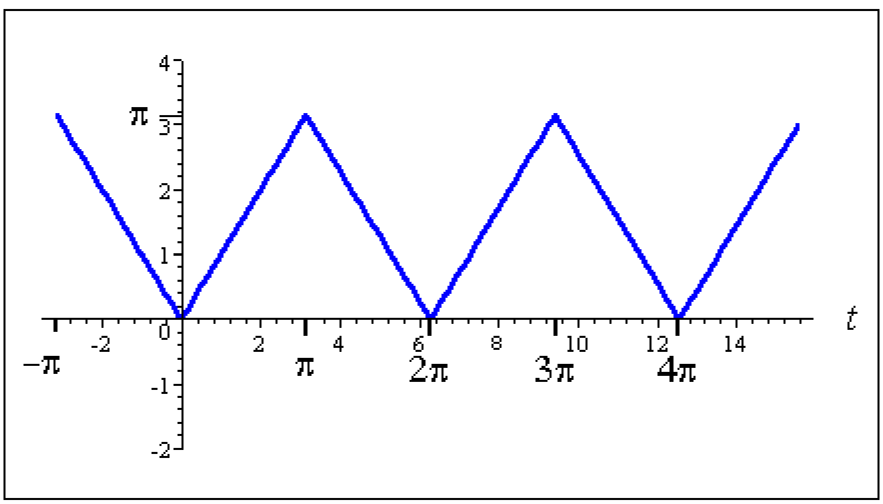
and $f(t)$ is periodic with period 2π.

a) Sketch the function for 3 cycles.

b) Find the Fourier trigonometric series for $f(t)$, using half-range series.

Answer

a) Sketch:



b) Since the function is even, we have $b_n = 0$.

In this example, $L = \pi$.

We have:

$$
\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(t)dt \\
&= \frac{2}{\pi}\int_0^\pi t\,dt \\
&= \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^\pi \\
&= \frac{2}{\pi}\frac{\pi^2}{2} \\
&= \pi
\end{aligned}
$$

To find a_n, we use a result from before:

$$\int t\cos nt\,dt = \frac{1}{n^2}(\cos nt + nt\sin nt)$$

We have:

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}dt \\
&= \frac{2}{\pi}\int_0^\pi t\;\cos nt\,dt \\
&= \frac{2}{\pi}\left[\frac{1}{n^2}(\cos nt + nt\sin nt)\right]_0^\pi \\
&= \frac{2}{\pi n^2}[(\cos n\pi + 0) - (\cos 0 + 0)] \\
&= \frac{2}{\pi n^2}[\cos n\pi - 1)] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1]
\end{aligned}
$$

When n is odd, the last line gives us $-\frac{4}{\pi n^2}$.

When n is even, the last line equals 0.

For the series, we need to generate odd values for n. We need to use $(2n - 1)$ for $n = 1, 2, 3, \ldots$

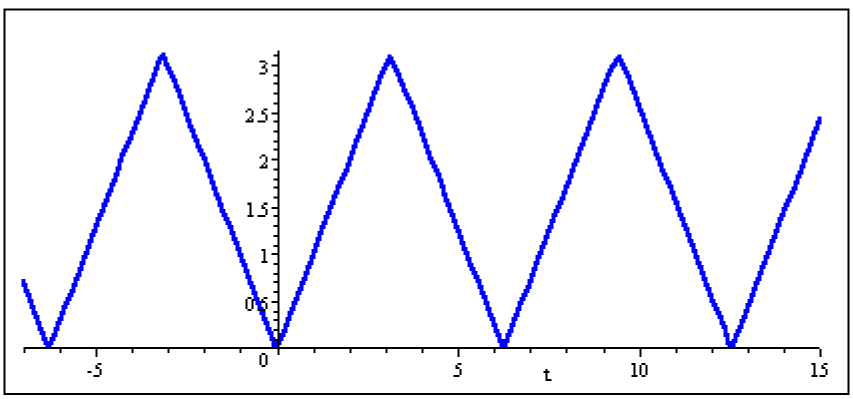
So we have:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)t}{(2n-1)^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos t + \frac{1}{9} \cos 3t + \frac{1}{25} \cos 5t + \ldots \right)$$

Check: The graph for the first 40 terms:



Odd Function and Half Range Sine Series

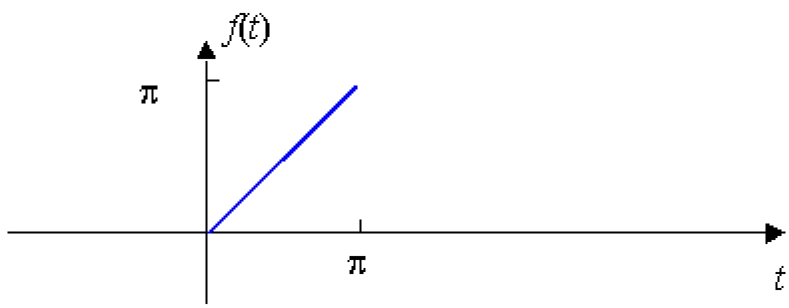
An odd function can be expanded using half its range from 0 to L, i.e. the range of integration = L. The Fourier series of the odd function is:

Since $a_o = 0$ and $a_n = 0$, we have:

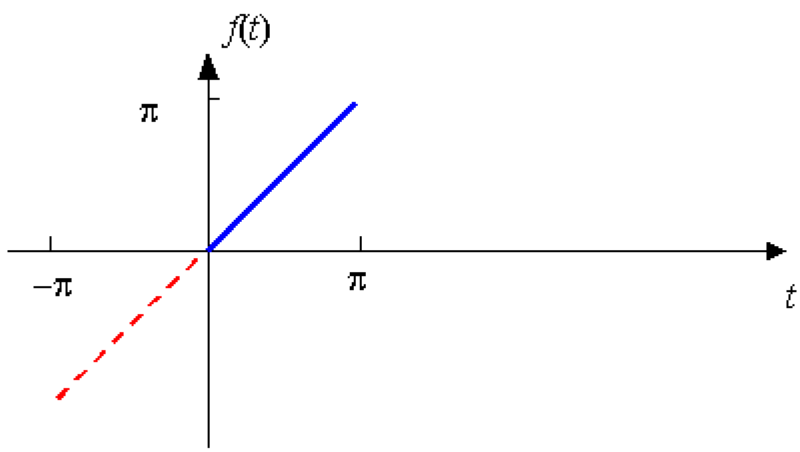
$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$ for $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt$$

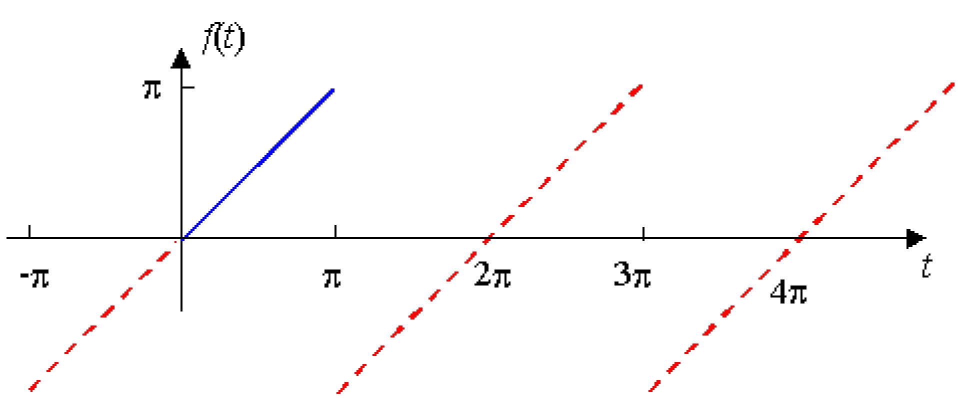
In the figure below, $f(t) = t$ is sketched from $t = 0$ to $t = \pi$, as before.



An **odd** function means that it is symmetrical about the origin and this is shown by the red broken lines between $t = -\pi$ and $t = 0$.



It is then assumed that the waveform produced is periodic of period 2π outside of this range as shown by the dotted lines.



5. Harmonic Analysis

Recall the Fourier series (that we met in Full Range Fourier Series):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \ldots$$

$$+ b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots$$

We can re-arrange this series and write it as:

$$f(t) = \frac{a_0}{2} + (a_1 \cos t + b_1 \sin t) + (a_2 \cos 2t + b_2 \sin 2t) + (a_3 \cos 3t + b_3 \sin 3t) + \ldots$$

The term $(a_1 \cos t + b_1 \sin t)$ is known as the **fundamental**.

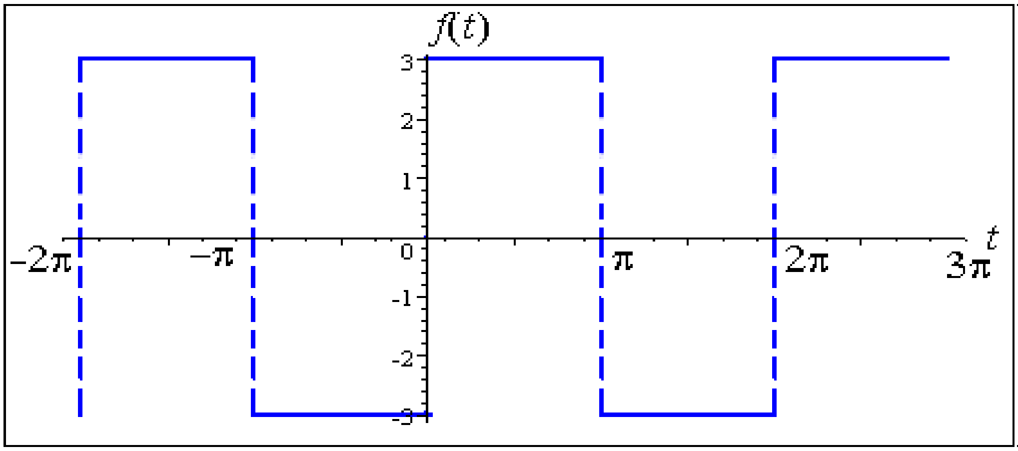
The term $(a_2 \cos 2t + b_2 \sin 2t)$ is called the **second harmonic.**

The term $(a_3 \cos 3t + b_3 \sin 3t)$ is called the **third harmonic,** etc.

Odd Harmonics

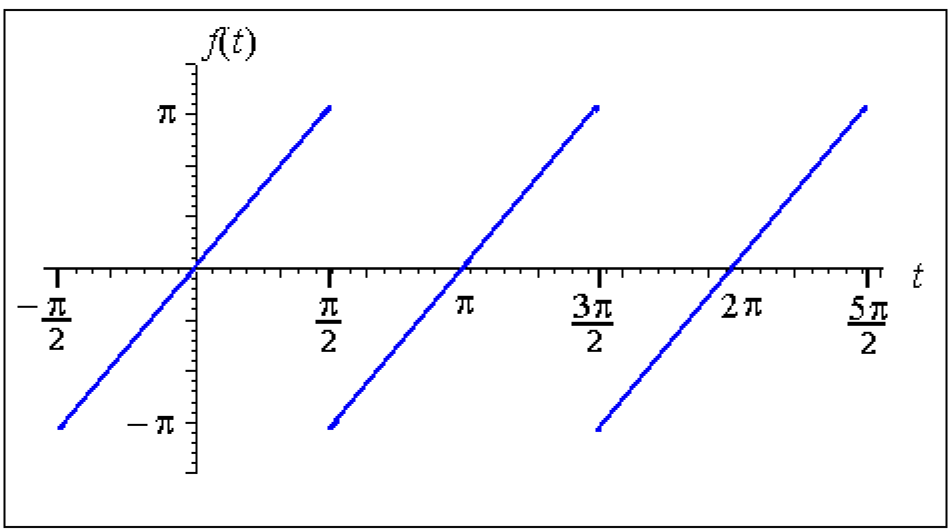
The Fourier series will contain **odd harmonics** if $f(t + \pi) = -f(t)$.

Example:



Even Harmonics

The Fourier series will contain **even harmonics** if $f(t + \pi) = f(t)$.
(That is, it has period π.)



EXAMPLES

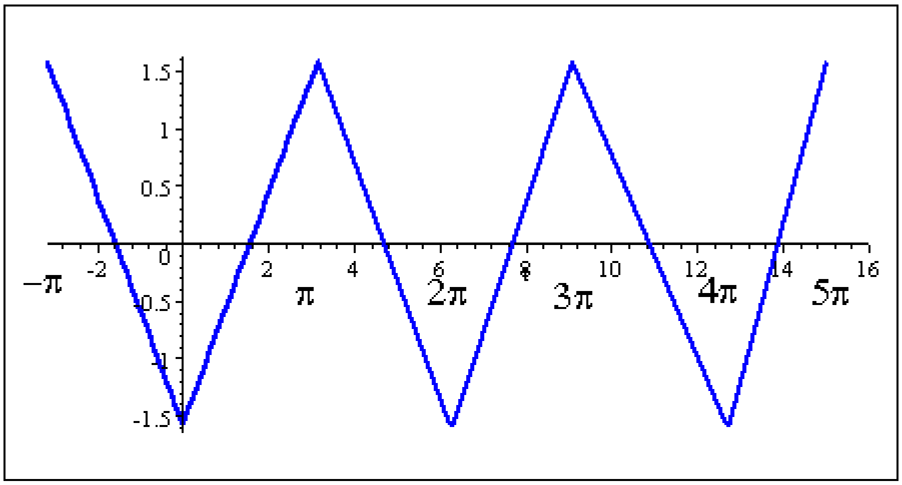
Determine the existence of odd or even harmonics for the following functions.

(a)

$$f(t) = \begin{cases} -t - \dfrac{\pi}{2} & \text{if } -\pi \le t < 0 \\ t - \dfrac{\pi}{2} & \text{if } 0 \le t < \pi \end{cases}$$

$f(t) = f(t + 2\pi)$.

Answer



We can see from the graph that $f(t + \pi) = -f(t)$.

For example, we notice that $f(2) = 0.4$, approximately. If we now move π units to the right (or about 2 + 3.14 = 5.14), we see that the function value is

$f(5.14) = -0.4$.

That is, $f(t + \pi) = -f(t)$.

This same behaviour will occur for any value of t that we choose.

So the Fourier Series will have **odd harmonics**.

6. Line Spectrum

Recall from earlier earlier trigonometry that we can express the sum of a sine term and a cosine term, with the same period, as follows:

$a \cos \theta + b \sin \theta = R \cos (\theta - \alpha)$

where

$$R = \sqrt{a^2 + b^2}$$

and

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

Likewise, the Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

can be written in **harmonics form**

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t - \phi_n)$$

where

$C_1 \cos (\omega t - \varphi_1)$ is called the **fundamental**

$C_2 \cos (2\omega t - \varphi_2)$, the **second harmonic**

$C_3 \cos (3\omega t - \varphi_3)$, the **third harmonic** ... etc; and

φ_n = phase angle

$$C_n = \text{harmonic amplitude} = \sqrt{a_n^2 + b_n^2}$$

We met harmonics before.

A plot showing each of the harmonic amplitudes in the wave is called the **line spectrum**.

Note: Waves with discontinuities such as the saw tooth and square wave have spectra with slowly decreasing amplitudes since their series have strong high harmonics. Their 10th harmonics will often have amplitudes of significant value compared to the fundamental.

In contrast, the series of waveforms without discontinuities and with a generally smooth appearance will converge rapidly to the function and only a few terms are required to generate the wave.

Let's see a Livemath animation of this.

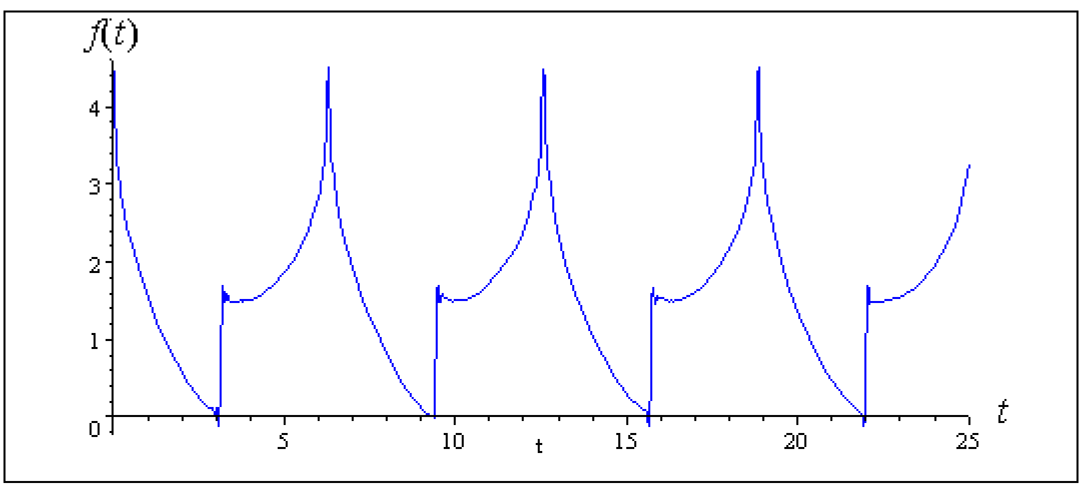
LIVEMath

EXAMPLE

Plot the line spectrum for the Fourier Series:

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos nt + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sin nt$$

This series has an interesting graph:



Answer

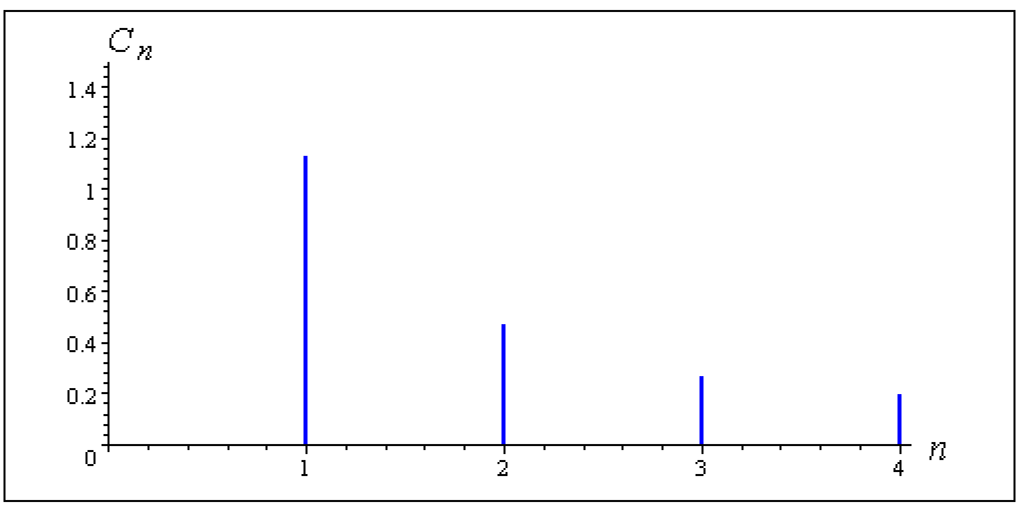
We can see from the series that

$$a_n = \frac{1}{2n-1} \quad b_n = \frac{(-1)^n}{2n}$$

Now, using $C_n = \sqrt{a_n^2 + b_n^2}$ for each term, we have:

$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2 + b_n^2}$
$a_1 = 1$	$b_1 = -\frac{1}{2}$	$C_1 = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = 1.118$
$a_2 = \frac{1}{3}$	$b_2 = \frac{1}{4}$	$C_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = 0.4167$
$a_3 = \frac{1}{5}$	$b_3 = -\frac{1}{6}$	$C_3 = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{6}\right)^2} = 0.260$
$a_4 = \frac{1}{7}$	$b_4 = \frac{1}{8}$	$C_4 = \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{1}{8}\right)^2} = 0.190$

The resulting line spectrum is:



Music Examples

1. Harmonics and Sound

When we listen to different musical instruments playing the same note, they sound different to us because of the different combinations of **harmonics** contained in the note.

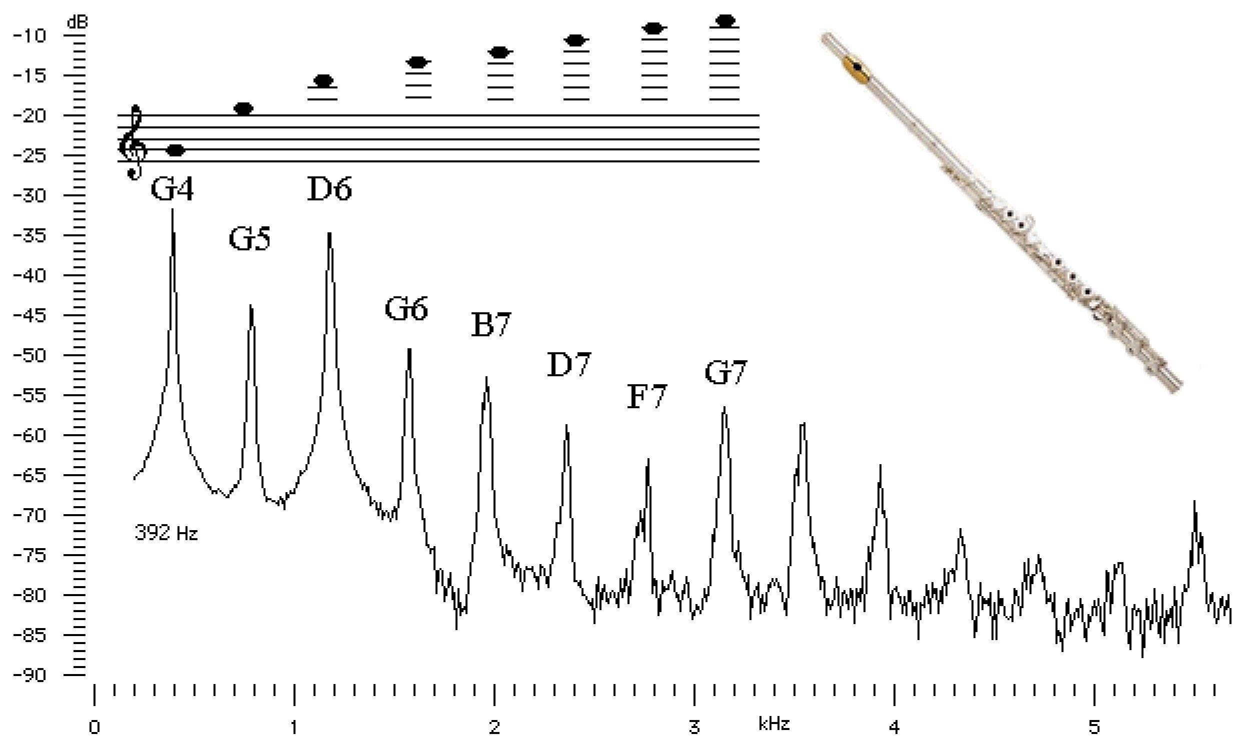
For example, if a flute and a violin both play G above middle C, the harmonic spectrum is quite different.

G has a frequency of 392 Hz and the harmonics are all multiples of this fundamental frequency (or about 800 Hz, 1200 Hz, 1600 Hz, etc).

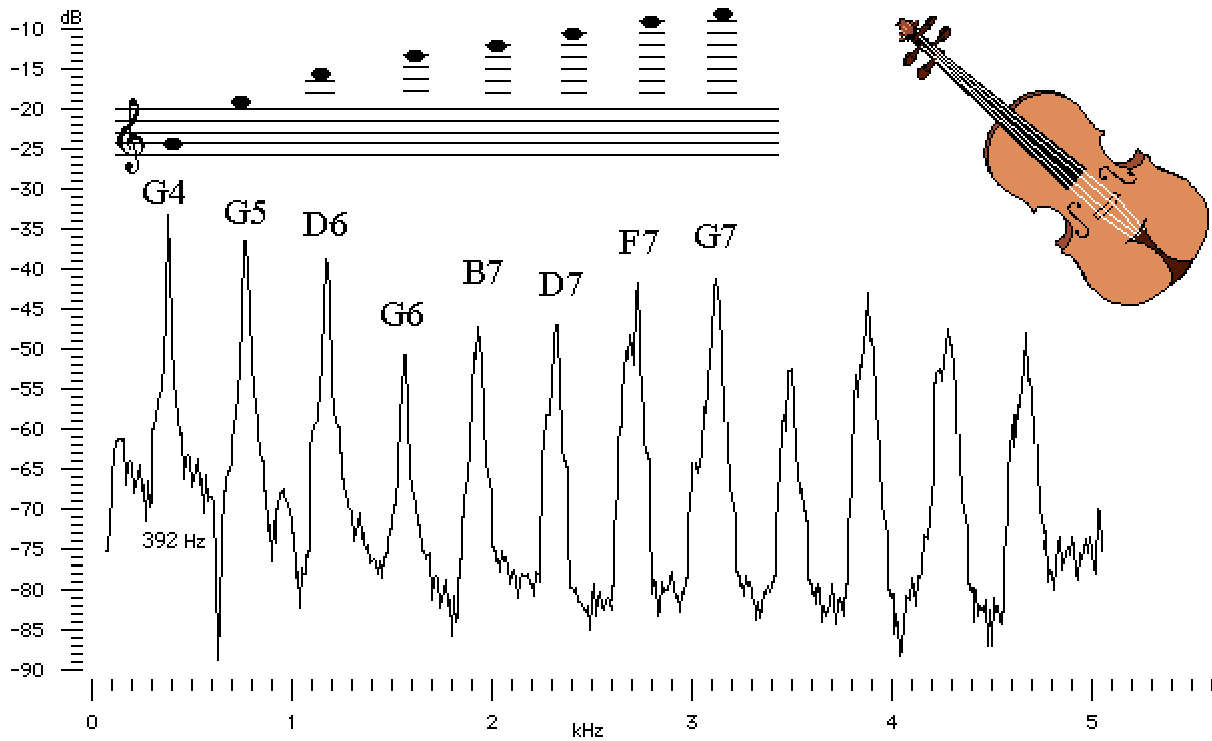
Harmony (2 or more notes sounding at the same time) works because of harmonics (look for the chord GBD contained within the harmonics of the note G).

You can see the relative values of the harmonics in the following sound spectrum images of a flute and a violin playing G4.

Flute



Violin



2. Java Applet - Fourier Series and Sound

The following link leads to a great Java applet to play with. (It's on an external site.) He is using Fourier Series expressed in the form

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t - \phi_n)$$

You can change the mean value (C_0), amplitude of each harmonic (C_n) and also f_n, which changes the phase shift. You can hear the sounds that result from adding different harmonics.

http://falstad.com/fourier/

While you're there, check out the other applets and lots of interesting stuff at:

http://falstad.com/mathphysics.html

7. Application - The Fast Fourier Transform

1. Digital Audio

Pulse code modulation (PCM) is the most common type of digital audio recording, used to make compact disks and WAV files.

In PCM recording hardware, a microphone converts sound waves into a varying voltage. Then an analog-to-digital converter samples the voltage at regular intervals of time. For example, in a compact disc audio recording, there are 44100 samples taken every second.

The data that results from a PCM recording is a function of time. How does this work?

Imagine that you were very small and could fit into your friend's ear drum. Suppose also that you could see things in very slow motion and that you could record the position of the ear drum once every 44100th of a second. Your eyes are so good that you can notice 65536 distinct positions of the ear drum's surface as it moves back and forth in response to incoming sound waves.

If your friend is listening to the sound of a flute, and you write down the positions of the ear drum that you notice, then you would have a digital PCM recording - a series of numbers.

If you could later make your own ear drum move back and forth in accordance with the thousands of numbers you had written down, you would hear the flute exactly as it originally sounded. We have gone from:

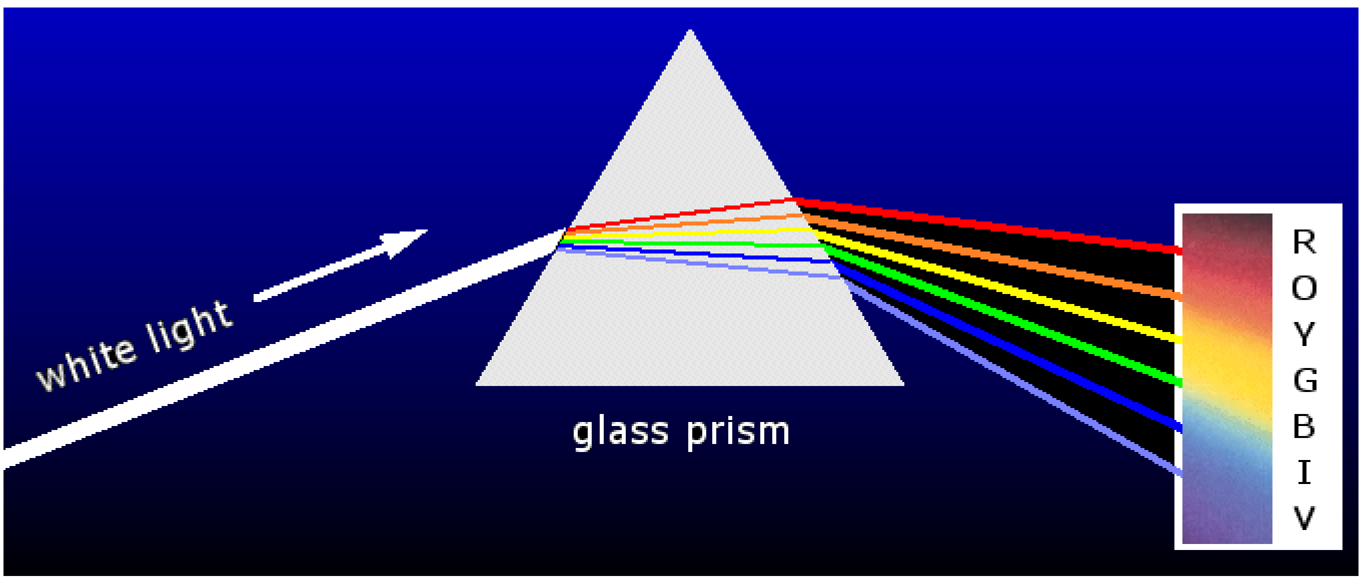
rich sound with fundamentals and harmonics
→ numbers
→ rich sound with fundamentals and harmonics

To be able to convert from the series of numbers to sound, we need to apply the **Fourier Transform**.

2. Frequency Information as a Function of Time

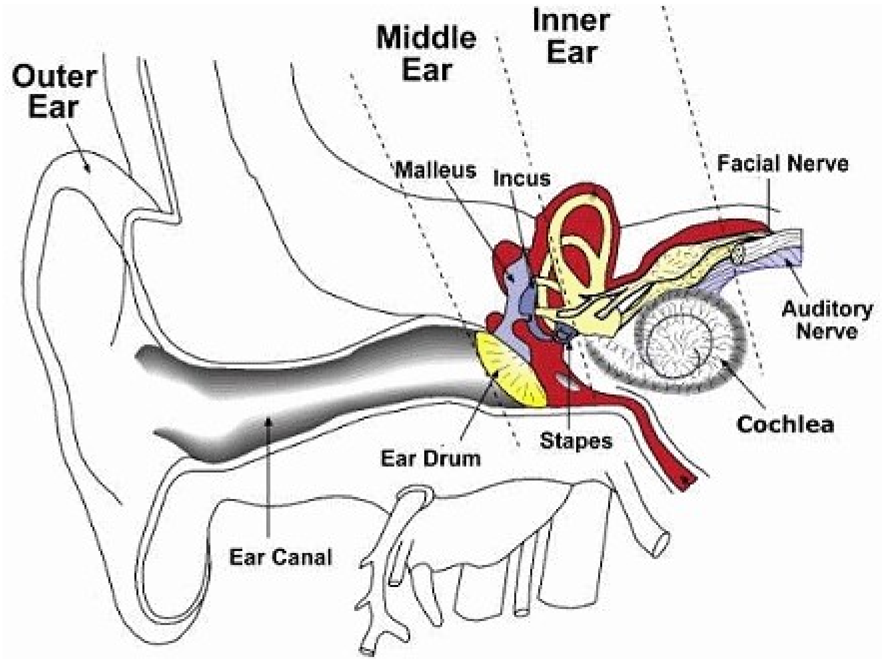
Prisms

One analogy for the type of thing a Fourier Transform does is a prism which splits white light into a spectrum of colors. White light consists of all frequencies mixed together (much like the information on a CD has sounds of all frequencies mixed together) and the prism breaks them apart so we can see the separate frequencies (much like the CD player splits apart the frequencies so that they can be amplified and sent to the speakers).



Cochlea

In our inner ears, the **cochlea** enables us to hear subtle differences in the sounds coming to our ears. The cochlea consists of a spiral of tissue filled with liquid and thousands of tiny hairs which gradually get smaller from the outside of the spiral to the inside. Each hair is connected to a nerve which feeds into the auditory nerve bundle going to the brain. The longer hairs resonate with lower frequency sounds, and the shorter hairs with higher frequencies. Thus the cochlea serves to transform the air pressure signal experienced by the ear drum into frequency information which can be interpreted by the brain as tonality and texture.



The **Fourier Transform** is a mathematical technique for doing a similar thing - resolving any time-domain function into a frequency spectrum. The **Fast Fourier Transform** is a method for doing this process very efficiently.

3. The Fourier Transform

As we saw earlier in this chapter, the Fourier Transform is based on the discovery that it is possible to take any **periodic** function of time $f(t)$ and resolve it into an equivalent infinite summation of sine waves and cosine waves with frequencies that start at 0 and increase in integer multiples of a base frequency $f_0 = 1/T$, where T is the period of $f(t)$. The resulting infinite series is called the **Fourier Series**:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

The job of a Fourier Transform is to figure out all the a_n and b_n values to produce a Fourier Series, given the base frequency and the function $f(t)$.

In our CD example, which has a sampling rate of 44100 samples/second, if the length of our recording is 1024 samples, then the amount of time represented by the recording is $\frac{1024}{44100} = 0.02322$ seconds, so the base frequency f_0 will be $\frac{1}{T} = \frac{44100}{1024} = 43.066$ Hz.

If you process these 1024 samples with the FFT (Fast Fourier Transform), the output will be the sine and cosine coefficients a_n and b_n for the frequencies

43. 066 Hz,
2 × 43. 066 = 86. 132 Hz,
3 × 43. 066 = 129. 20 Hz, etc.

Example

Let's say that we use the FFT to process a series of numbers on a CD, into a sound.

It may give us something like $a_0 = 1.6$, $a_n = \frac{(-1)^n}{n}$ and $b_n = \frac{1}{2n-1}$.

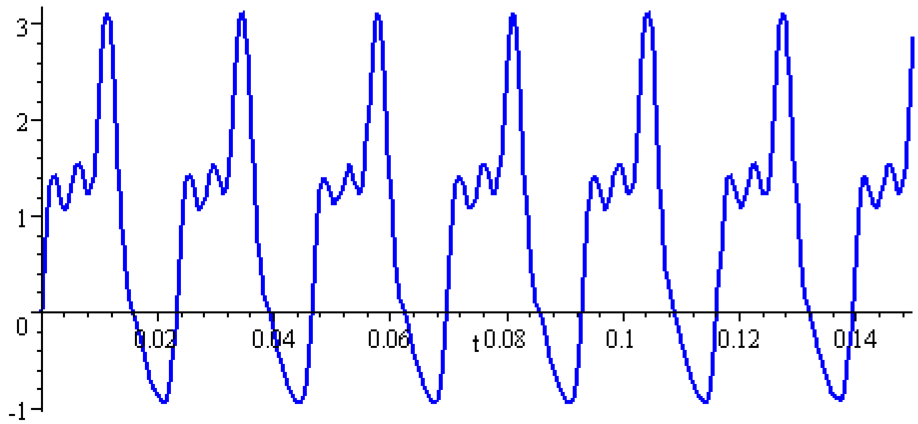
For the frequencies 43.066 Hz, 86.123 Hz and 129.20 Hz, we use $T = \frac{2\pi}{b}$, and we have:

$$b = \frac{44100(2\pi)}{1024} = 270.\ 59$$

So the Fourier Series would be:

$$f(t) = \frac{1.6}{2} + \sum_{n=1}^{5}\left(\frac{(-1)^n}{n}\cos 270.\ 59nt + \frac{1}{2n-1}\sin 270.\ 59nt\right)$$

$= -1.0\cos 270.59t + \sin 270.59t + 0.5\cos 541.18t + 0.333\,33\sin 541.18t -$
$0.33333\cos 811.77t + 0.2\sin 811.77t + 0.25\cos 1082.4t + 0.14286\sin 1082.4t$
$-0.2\cos 1353.0t + 0.11111\sin 1353.0t$



We have reconstructed a sound wave from the digital data fed from the CD into the sound system of the CD player.

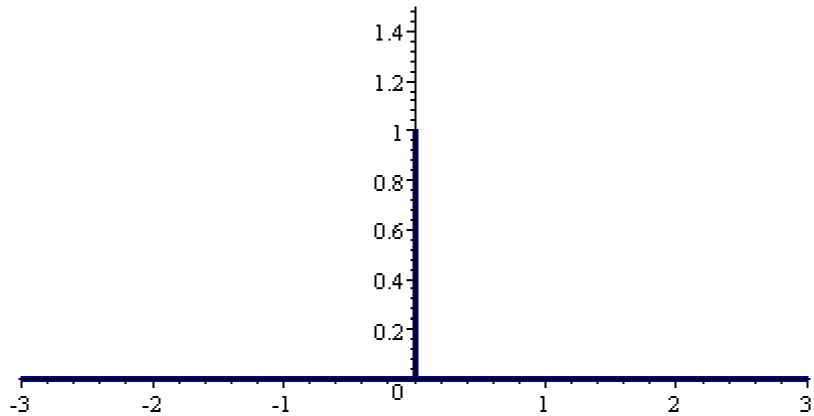
The Fourier Transform Formula

If *f* is a real-valued function on $[-\infty, \infty]$, the function $\hat{f} = \mathcal{F}(f)$ defined by the integral

$$\mathcal{F}(f(t), x, w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt$$

is the Fourier Transform of the function *f*.

Fourier Transforms involve the Dirac (or delta, δ) function (also known as the **pulse** function) which has magnitude 1 at $t = 0$, but is 0 elsewhere.



Dirac(x)

This delta function is beyond the scope of this chapter.
